Teacher notes
Topic B

Fun with emissivity

All bodies kept at some kelvin temperature *T* radiate. A theoretical body called a black body radiates an intensity (radiated power per unit area) equal to $I=\sigma T^4$. All other bodies radiate an intensity $I=e\sigma T^4$ where the constant *e* is called the emissivity of the surface. It is a dimensionless number that varies from 0 to 1. The emissivity of a body is therefore the ratio

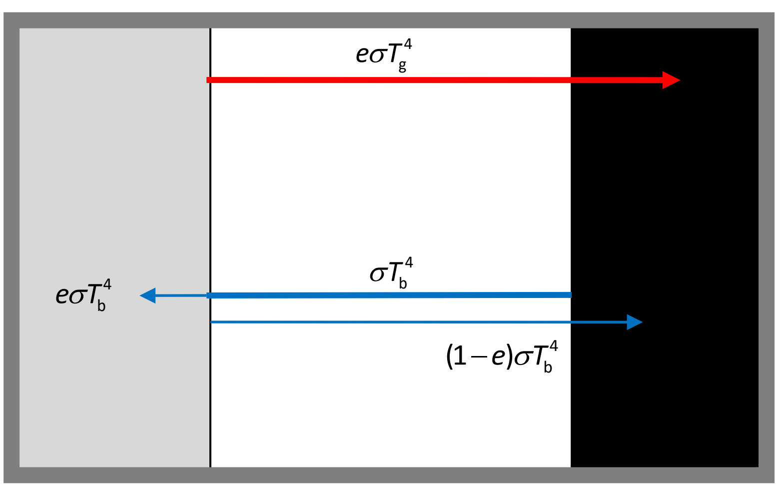
$$\frac{\text{intensity radiated by body}}{\text{intensity radiated by black body at same temperature}}.$$

A black body will absorb all the radiation incident on it reflecting none. A gray body of emissivity *e* will absorb a fraction *e* of the incident intensity on it reflecting a fraction (1 – *e*).

Suppose then that we have a gray body of emissivity *e* and a black body nearby.

The two bodies exchange radiation with no radiation going anywhere else other than between the two bodies. The gray body is kept at temperature T_g and the black body at a lower temperature T_b.

What is the net intensity leaving the gray body?



An intensity $e\sigma T_g^4$ is radiated by the gray body and an intensity σT_b^4 is radiated by the black body. Of the black body radiation incident on the gray body a fraction *e* is absorbed. Thus:

Net intensity *leaving* gray body is: $e\sigma T_g^4 - e\sigma T_b^4 = e\sigma(T_g^4 - T_b^4)$.

An intensity σT_b^4 leaves the black body. A fraction *e* is absorbed by the gray body and a fraction $1-e$ is reflected by the gray body. All of the reflected intensity is absorbed by the black body. All the intensity $e\sigma T_g^4$ radiated by the gray body incident on the black body is absorbed. Thus,

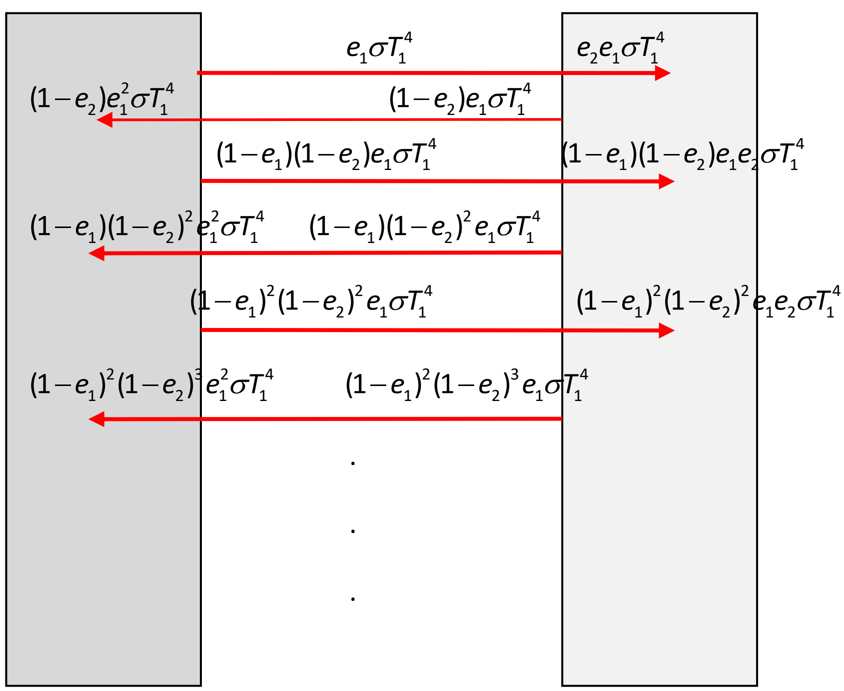
Net intensity *leaving* black body: $\sigma T_b^4 - (1-e)\sigma T_b^4 - e\sigma T_g^4 = e\sigma(T_b^4 - T_g^4)$.

At equilibrium the net intensity leaving each body must be zero and hence, in that case, the temperatures must be the same.

A more interesting problem arises when both bodies are gray with emissivities e_1 and e_2. Again, we assume that the only exchanges of energy are between the 2 bodies and nothing else.

This requires some proficiency with algebra and might be appreciated by Math HL students.

The diagram shows the radiation leaving the left body (of temperature T_1), absorbed by the body on the right, reflected by the body on the right and then absorbed and reflected by the body on the left. This goes on forever. The diagram does not show the intensity radiated by the body on the right.



An intensity $e_1\sigma T_1^4$ is radiated by the left body and so this intensity leaving the body.

Radiation enters the left body in 2 ways:

Part of the radiated $e_1\sigma T_1^4$ gets reflected by the body on the right and part of this reflected radiation enters the body. The reflected part again gets reflected from the right body and again part of this enters the left body. This is an infinite process.

The total intensity entering body on left in this way is

$$
\begin{aligned}
&(1-e_2)e_1^2\sigma T_1^4 + (1-e_1)(1-e_2)^2 e_1^2\sigma T_1^4 + (1-e_1)^2(1-e_2)^3 e_1^2\sigma T_1^4 + \ldots \\
&= (1-e_2)e_1^2\sigma T_1^4\left(1+(1-e_1)(1-e_2)+(1-e_1)^2(1-e_2)^2+\ldots\right)
\end{aligned}
$$

The quantity in big brackets is an infinite geometric series.

The second way radiation enters the body on the left is exactly the method of the first way but now applied to the radiation emitted by the body on the right. Since our diagram shows the radiation entering the right body all we have to do is look at that intensity but switch $1 \leftrightarrow 2$ in order to get the radiation entering the left body. The result is:

$$
\begin{aligned}
&e_1e_2\sigma T_2^4 + (1-e_1)(1-e_2)e_1e_2\sigma T_2^4 + (1-e_1)^2(1-e_2)^2 e_1e_2\sigma T_2^4 + \ldots \\
&= e_1e_2\sigma T_2^4\left(1+(1-e_1)(1-e_2)+(1-e_1)^2(1-e_2)^2+\ldots\right)
\end{aligned}
$$

The total intensity entering the left body is then

$$
\begin{aligned}
&= (1-e_2)e_1^2\sigma T_1^4\left((1+(1-e_1)(1-e_2)+(1-e_1)^2(1-e_2)^2+\ldots\right) + e_1e_2\sigma T_2^4\left(1+(1-e_1)(1-e_2)+(1-e_1)^2(1-e_2)^2+\ldots\right) \\
&= ((1-e_2)e_1^2\sigma T_1^4 + e_1e_2\sigma T_2^4)\left((1+(1-e_1)(1-e_2)+(1-e_1)^2(1-e_2)^2+\ldots\right) \\
&= ((1-e_2)e_1^2\sigma T_1^4 + e_1e_2\sigma T_2^4)\frac{1}{1-(1-e_1)(1-e_2)} \qquad \text{(summed an infinite geometric series)} \\
&= \frac{e_1^2(1-e_2)\sigma T_1^4 + e_1e_2\sigma T_2^4}{e_1+e_2-e_1e_2}
\end{aligned}
$$

The net intensity leaving the left body is then

$$
\begin{aligned}
I_{\text{net}} &= e_1\sigma T_1^4 - \frac{e_1^2(1-e_2)\sigma T_1^4 + e_1e_2\sigma T_2^4}{e_1+e_2-e_1e_2} \\
I_{\text{net}} &= \frac{e_1^2\sigma T_1^4 + e_1e_2\sigma T_1^4 - e_1^2e_2\sigma T_1^4 - e_1^2\sigma T_1^4 + e_1^2e_2\sigma T_1^4 - e_1e_2\sigma T_2^4}{e_1+e_2-e_1e_2} \\
I_{\text{net}} &= \frac{e_1e_2\sigma T_1^4 - e_1e_2\sigma T_2^4}{e_1+e_2-e_1e_2}
\end{aligned}
$$

i.e.

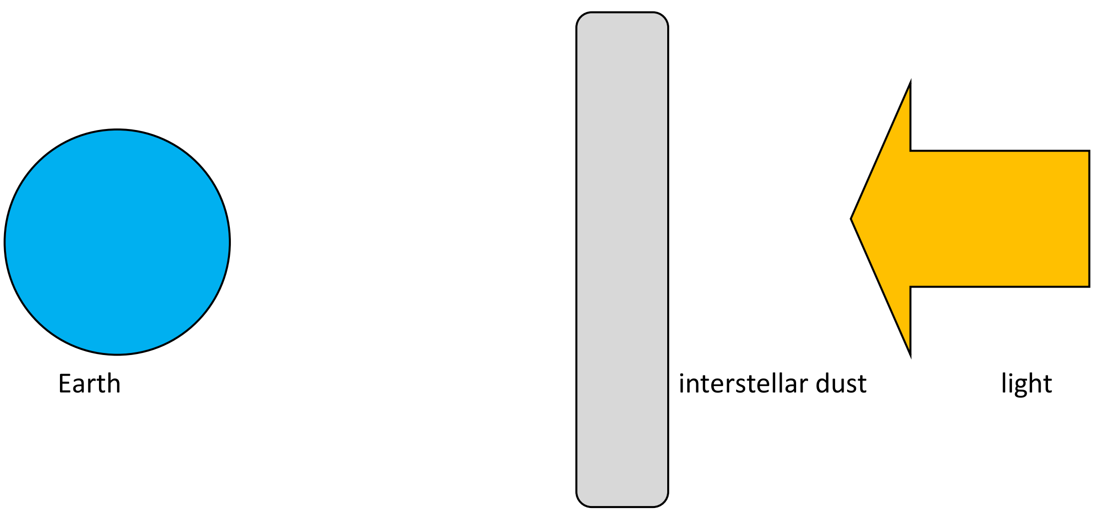
$$I_{\text{net}} = \frac{e_1 e_2}{e_1 + e_2 - e_1 e_2}\sigma(T_1^4 - T_2^4)$$

If the right body is a black body (as in our starting example) then (with $e_2 = 1$)

$I_{\text{net}} = \frac{e_1 \times 1}{e_1 + 1 - e_1 \times 1}\sigma(T_1^4 - T_2^4) = e_1\sigma(T_1^4 - T_2^4)$ as we found before.

Olbers' "paradox"

In the Newtonian view of the world the Universe is an infinite unchanging collection of stars and galaxies all radiating energy away. In 1823, Olbers (and many others before him, Kepler, Halley, Cheseaux) noticed that if this is the case, then no matter where you looked at in the sky your line of sight would end on a star. The night sky would then never be black, it would be bright, at least as bright as a night with a full moon. An early objection to Olbers was that interstellar dust, material in between stars, would absorb some of the light and so the light reaching Earth would be much reduced, making the night sky dark.



The previous discussion invalidates the objection to Olbers. In the infinite time available in Newton's Universe the energy absorbed by the interstellar dust would increase its temperature to that of the sources of light incident on the dust. Thus, the interstellar dust would radiate the same intensity as that of the original sources and this radiation arriving at Earth would make for a bright sky.

Olbers' "paradox" is solved by realizing that the Universe is finite, so it contains a finite number of stars, the stars do not live forever and in addition the Universe is expanding so that the radiation reaching Earth is redshifted making the incident intensity smaller.

Thus, the very simple question of “why is the night sky dark?” led to an abandonment of the Newtonian view of an infinite, static and ageless Universe.